Surface to surface missile on the moon

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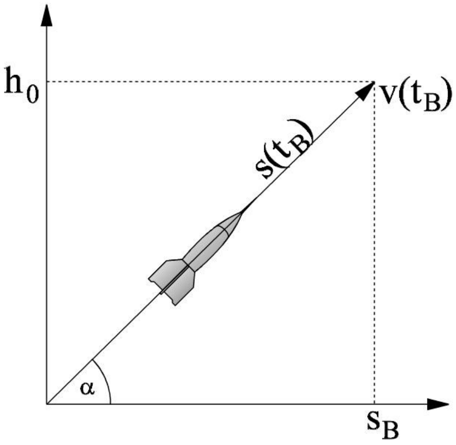
We want to use our rocket-propelled moon rover now not only as a ground vehicle but also as a surface to surface missile. Therefore we are interested for which launch angle we get the greatest horizontal distance and which maximum altitude we will reach.
The horizontal distance s of a parabolic trajectory in a uniform gravitational field and in the absence of other forces is given by

$$(0.1)\qquad s=\frac{v_0\cdot\cos(\alpha)}{g}\cdot\left(v_0\cdot\sin(\alpha)+\sqrt{v_0^2\cdot\sin^2(\alpha)+2\cdot g\cdot h_0}\right)$$

with the initial velocity v_0, the initial angle α, the initial height h_0 and the acceleration of gravity g.
The rocket shall be launched from the completely flat lunar surface with the initial angle α. We further assume the trajectory till burn-out $s(t_B)$ is a straight line:



We get

$$(0.2)\qquad \sin(\alpha)\cdot s(t_B)=h_0$$

$$(0.3)\qquad \cos(\alpha)\cdot s(t_B)=s_B$$

The initial velocity v_0 is the burn-out velocity $v(t_B)$, and thus the total horizontal distance s_t of our lunar surface to surface missile is:

(0.4)

$$s_t = \frac{v(t_B)\cdot\cos(\alpha)}{g}\cdot\left(v(t_B)\cdot\sin(\alpha)+\sqrt{v(t_B)^2\cdot\sin^2(\alpha)+2\cdot g\cdot\sin(\alpha)\cdot s(t_B)}\right)+\cos(\alpha)\cdot s(t_B)$$

A necessary condition for $f(x)$ to have a local maximum at x_0 is

$f'(x_0)=0$

A sufficient condition to have a local maximum at x_0 is

$f'(x_0)=0$ and $f''(x_0)<0$

Thus

(0.5)
$$s_t'(\alpha) = -\left(\frac{\sin(\alpha)\cdot v(t_B)^2+\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}\cdot v(t_B)+g\cdot s(t_B)}{g\cdot\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}}\right)\cdot$$
$$\left(\sin(\alpha)\cdot\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}-v(t_B)\cdot\cos^2(\alpha)\right)$$

(0.6) $s_t'(\alpha)=0$

(0.7) $\sin(\alpha)\cdot v(t_B)^2+\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}\cdot v(t_B)+g\cdot s(t_B)=0$

(0.8) $\sin(\alpha)\cdot\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}-v(t_B)\cdot\cos^2(\alpha)=0$

Simplifying 0.7:

(0.9) $\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}\cdot v(t_B)=-\sin(\alpha)\cdot v(t_B)^2-g\cdot s(t_B)$

(0.10)

$$\sin^2(\alpha)\cdot v(t_B)^4+\sin(\alpha)\cdot 2\cdot g\cdot s(t_B)\cdot v(t_B)^2=\sin^2(\alpha)\cdot v(t_B)^4+2\cdot g\cdot s(t_B)\cdot\sin(\alpha)\cdot v(t_B)^2+g^2\cdot s(t_B)^2$$

(0.11) $g^2\cdot s(t_B)^2=0$

Either g nor $s(t_B)$ can be 0, so we have to find the roots in 0.8

Simplifying 0.8 by using the trigonometric definition

(0.12) $\sin^2(\alpha)+\cos^2(\alpha)=1$

(0.13) $\Leftrightarrow \cos^2(\alpha)=1-\sin^2(\alpha)$

(0.14) $\sin(\alpha)\cdot\sqrt{\sin(\alpha)\cdot\left(\sin(\alpha)\cdot v(t_B)^2+2\cdot g\cdot s(t_B)\right)}-v(t_B)\cdot\left(1-\sin^2(\alpha)\right)=0$

(0.15) $2\cdot g\cdot s(t_B)\cdot\sin^3(\alpha)=v(t_B)^2\cdot\left(1-2\cdot\sin^2(\alpha)\right)$

(0.16) $\sin^3(\alpha)+\frac{v(t_B)^2}{g\cdot s(t_B)}\cdot\sin^2(\alpha)-\frac{v(t_B)^2}{2\cdot g\cdot s(t_B)}=0$

The initial angle α must be greater than 0º and smaller than 90º, thus $\sin(\alpha)\in\left]0,1\right[$.
We want now to know, how many positive roots the cubic equation 0.16 for $\frac{v(t_B)^2}{g\cdot s(t_B)}\in\mathbb{R}_{>0}$ has.

Substituting $\sin(\alpha)=x$

(0.17) $x^3+\frac{v(t_B)^2}{g\cdot s(t_B)}\cdot x^2-\frac{v(t_B)^2}{2\cdot g\cdot s(t_B)}=0$

Studying the extrema of $f(x)$:

(0.18) $f(x)=x^3+\frac{v(t_B)^2}{g\cdot s(t_B)}\cdot x^2-\frac{v(t_B)^2}{2\cdot g\cdot s(t_B)}$

(0.19) $f'(x)=3\cdot x^2+\frac{2\cdot v(t_B)^2}{g\cdot s(t_B)}\cdot x$

(0.20) $f'(x)=0$

(0.21) $3\cdot x^2+\frac{2\cdot v(t_B)^2}{g\cdot s(t_B)}\cdot x=0$

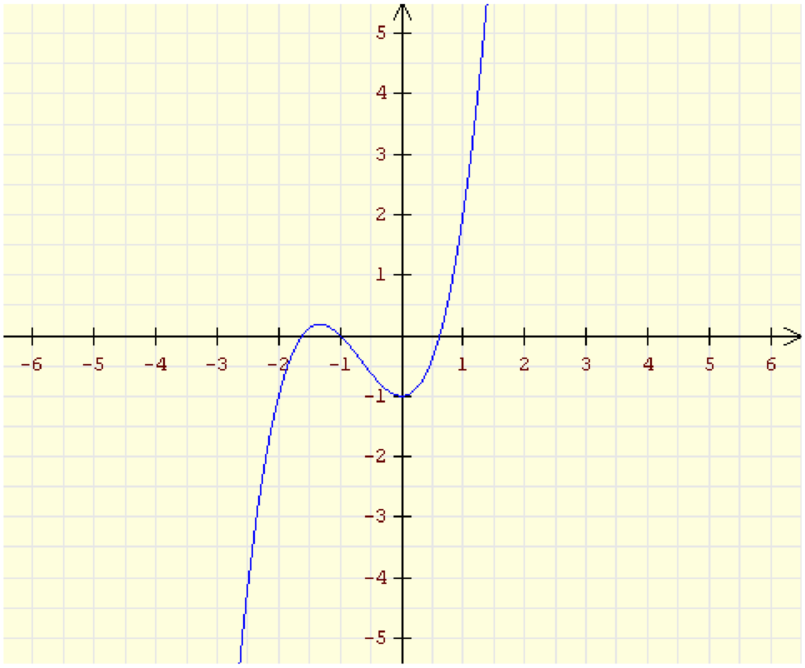
We get two roots: $x_1=0$ and $x_2=-\frac{2\cdot v(t_B)^2}{3\cdot g\cdot s(t_B)}$

(0.22) $f''(x) = 6 \cdot x + \dfrac{2 \cdot v(t_B)^2}{g \cdot s(t_B)}$

(0.23) $f''(0) > 0 \Rightarrow f(x)$ has a local minimum at the point $\left(0 \middle| -\dfrac{v(t_B)^2}{2 \cdot g \cdot s(t_B)}\right)$

(0.24) $f''(-\dfrac{2 \cdot v(t_B)^2}{3 \cdot g \cdot s(t_B)}) < 0 \Rightarrow f(x)$ has a local maximum at the point $\left(-\dfrac{2 \cdot v(t_B)^2}{3 \cdot g \cdot s(t_B)} \middle| \dfrac{4 \cdot v(t_B)^6}{27 \cdot g^3 \cdot s(t_B)^3} - \dfrac{v(t_B)^2}{2 \cdot g \cdot s(t_B)}\right)$.

We know the cubic polynomial can have at most three real roots. This cubic polynomial has the maximum point at $x = -\dfrac{2 \cdot v(t_B)^2}{3 \cdot g \cdot s(t_B)}$ and the minimum point at $x = 0$. Therefore the graph of this cubic polynomial crosses the x-axis at a positive value exactly one time:



Because

(0.25) $\dfrac{x^3}{\left(\dfrac{1}{2} - x^2\right)} = \dfrac{v(t_B)^2}{g \cdot s(t_B)}$

(0.26) $\lim\limits_{x \to 0} \dfrac{x^3}{\left(\frac{1}{2} - x^2\right)} = 0$

and

(0.27) $\lim\limits_{x \xrightarrow{>0} \sqrt{\frac{1}{2}}} \dfrac{x^3}{\left(\frac{1}{2} - x^2\right)} = \infty$

our positive root x_+ must be $\in \left]0, \sqrt{\frac{1}{2}}\right[$, thus $\sin(\alpha)_+ \in \left]0, \sqrt{\frac{1}{2}}\right[\Leftrightarrow \alpha \in \left]0^o, 45^o\right[$.

The cubic equation has a closed-form solution. To find the roots of the cubic equation a free online cubic equation calculator can be used [1].

Now we prove the sufficient condition. The second derivative of 0.4 is:

(0.28)

$$s_t''(\alpha) = -\frac{1}{\sqrt{\sin(\alpha)} \cdot \sqrt{v(t_B)^2 \cdot \sin(\alpha) + 2 \cdot g \cdot s(t_B)} \cdot \left(g \cdot v(t_B)^2 \cdot \sin^2(\alpha) + 2 \cdot g^2 \cdot s(t_B) \cdot \sin(\alpha)\right)} \cdot$$

$$\left(4 \cdot v(t_B)^5 \cdot \cos(\alpha) \cdot \sin^4(\alpha) + \sqrt{\sin(\alpha)} \cdot \sqrt{v(t_B)^2 \cdot \sin(\alpha) + 2 \cdot g \cdot s(t_B)} \cdot \left(4 \cdot v(t_B)^4 \cdot \cos(\alpha) \cdot \sin^3(\alpha) + \right.\right.$$
$$\left. 9 \cdot g \cdot s(t_B) \cdot v(t_B)^2 \cdot \cos(\alpha) \cdot \sin^2(\alpha) + 2 \cdot g^2 \cdot s(t_B)^2 \cdot \cos(\alpha) \cdot \sin(\alpha)\right) +$$
$$13 \cdot g \cdot s(t_B) \cdot v(t_B)^3 \cdot \cos(\alpha) \cdot \sin^3(\alpha) + 10 \cdot g^2 \cdot s(t_B)^2 \cdot v(t_B) \cdot \cos(\alpha) \cdot \sin^2(\alpha) +$$
$$\left. g^2 \cdot s(t_B)^2 \cdot v(t_B) \cdot \cos^3(\alpha)\right)$$

Because $\alpha_+ \in \left]0^o, 45^o\right[$:

(0.29) $s''(\alpha_+) < 0$

Thus we have a local maximum point at α_+.

Finally we consider the maximum altitude. The maximum altitude $h_{\max}$ of a parabolic trajectory in a uniform gravitational field and in the absence of other forces is given by

(0.30) $h_{\max} = h_0 + \dfrac{v_0{}^2 \cdot \sin^2(\alpha)}{2 \cdot g}$

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[1] http://easycalculation.com/algebra/cubic-equation.php

In our case $h_0 = \sin(\alpha)\cdot s(t_B)$, $v_0 = v(t_B)$ and $\alpha = \alpha_+$. Thus

$$(0.31)\quad h_{\max} = \sin(\alpha_+)\cdot s(t_B) + \frac{v(t_B)^2\cdot\sin^2(\alpha_+)}{2\cdot g}$$

We calculate an example. We use for our surface to surface missile the same parameters like for our rocket-propelled ground vehicle:

Initial total mass of the surface to surface missile $m_A = 0.52$ kg

Propellant mass $m_P = 0.0105$ kg

Final total mass $m_E = 0.5095$ kg

Burn time $t_B = 2.87$ s

Effective exhaust velocity $v_{rel} = 1765\ \frac{m}{s}$

Gravitational acceleration on the lunar surface $g = 1.62\ \frac{m}{s^2}$

$$(0.32)\quad v(t_B) = v_{rel}\cdot\ln\left(\frac{m_A}{m_E}\right) - g\cdot t_B \approx 31.35\ \frac{m}{s}$$

$$(0.33)\quad s(t_B) = v_{rel}\cdot t_B - \frac{v_{rel}\cdot m_E\cdot t_B\cdot\ln\left(\frac{m_A}{m_E}\right)}{m_A - m_E} - \frac{g\cdot t_B^2}{2} \approx 44.82\ \text{m}$$

$$(0.34)\quad \sin^3(\alpha) + 13.54\cdot\sin^2(\alpha) - 6.77 = 0$$

$\sin(\alpha_+) \approx 0.6897561872415725$

$\alpha_+ \approx 43.61^o$

$s_t \approx 639$ m

$h_{\max} \approx 175$ m